Abstract

To improve the performance of vehicle dynamics control systems, it is important to be able to estimate the friction force characteristics between the tires and the road. In this paper, we estimate the radius of the tire friction circle by using the relationship between the Self-Aligning Torque (SAT), and the lateral and longitudinal forces acting on each tire. Then, we propose a vehicle dynamics control system for four-wheel distributed traction/braking and four-wheel distributed steering that is based on an on-line nonlinear optimization algorithm that minimizes the maximum μ rate of the four tires by using the estimation of the radius of the tire friction circle.

Keywords
Vehicle dynamics control, Self aligning torque, Vehicle dynamics integrated management, Tire friction circle
1. Introduction

In turns with low lateral accelerations, the characteristics of Self-Aligning Torque (SAT) become more saturated than those of the lateral force, such that an algorithm for estimating the tire characteristics based on the relationship between SAT and the lateral force has been developed.1, 2)

While such methods can be applied to pure lateral slip motion, we clarified the relationship between the tire grip margin and SAT with combined lateral and longitudinal slip by using a brush model3, 4) and are proposing a method for estimating the radius of the tire friction circle with combined lateral and longitudinal slip.5) Furthermore, we are proposing a vehicle dynamics control system for four-wheel distributed steering and four-wheel distributed traction/braking systems, that aims to optimize the tire friction circle of each wheel. Then, the effects of the vehicle dynamics control system on four-wheel distributed steering and four-wheel distributed traction/braking systems are shown by simulations.

2. Estimation of tire friction circle

Each tire grip margin is estimated from the SAT, longitudinal and lateral force of each individual tire.5) The tire grip margin \( \epsilon \), normal SAT \( T_{SAT} \) and SAT model rate \( \gamma_{SAT} \) are defined as

\[
\epsilon = 1 - \sqrt{F_x^2 + F_y^2} / F, \quad \text{.................................(1)}
\]

\[
T_{SAT} = \left[ \frac{1}{6} + \frac{2l}{3} \frac{F_x}{K_B} \right] F_y, \quad \text{.................................(2)}
\]

\[
\gamma = \frac{T_{SAT}}{T_{SAT0}}, \quad \text{.................................(3)}
\]

where, \( l \) : ground contact length, \( K_B \) : cornering stiffness, \( F \) : radius of tire friction circle, \( F_x, F_y \) : longitudinal and lateral force, and \( T_{SAT0} \) : self-aligning torque (SAT). By using the brush model, the relationship between the SAT model rate, longitudinal force, and tire grip margin can be derived as

\[
\left( \frac{1}{6} + \frac{2}{3} \frac{F_x}{K_B} \right) \gamma_{SAT} (1 + \epsilon^{\alpha} + \epsilon^{2\alpha})^2
\]

\[
= \frac{1}{2} \epsilon \left( 1 + \epsilon^{\alpha} + \epsilon^{2\alpha} \right) + \frac{3}{5} \frac{F_x}{K_B} (1 + 2\epsilon^{\alpha} + 3\epsilon^{2\alpha} + 4\epsilon)^2.
\]

\[
\text{.................................(4)}
\]

Figure 1 shows the tire grip margin \( \epsilon \) as a function of the SAT model rate \( \gamma_{SAT} \) and normalized longitudinal force \( F_x/K_B \), as calculated from Eq. (4). This figure shows that the tire grip margin can be described by a monotonous function of the SAT model rate and the normalized longitudinal force. Then, the tire grip margin can be estimated from \( \gamma_{SAT} \) and \( F_x/K_B \) by using a three-dimensional map as shown in Fig. 1.

Furthermore, the radius of the tire friction circle \( F_0 \) can be estimated as:

\[
F = \sqrt{F_x^2 + F_y^2} / (1 - \epsilon) \quad \text{.................................(5)}
\]

3. Vehicle dynamics control for four-wheel distributed steering and traction/braking system

3.1 Hierarchical control structure

The hierarchical control structure shown in Fig. 2 is adopted for vehicle dynamics control.6)

[Vehicle Dynamics Control]

This layer calculates the target force and moment of the vehicle in order to achieve the desired vehicle motion corresponding to the driver's pedal input and the steering wheel angle. The designed target force and moment also satisfy the robust stability condition.7)

[Force & Moment Distribution]

The target force and moment of the vehicle motion are distributed to the target forces of each wheel, based on the tire grip margin in this layer.

Fig. 1 Relationship between \( \gamma_{SAT}, F_x/K_B \) and \( \epsilon \).
[Wheel Control]
This layer controls the motion of each wheel so as to achieve the target force.
This paper describes a force & moment distribution algorithm in detail.

3.2 Nonlinear optimal control
A vehicle model is described using the coordinates shown in Fig. 3, with the X-axis being the direction of the target resultant force and the Y-axis being the direction perpendicular to the X-axis. The μ rate $\gamma_i$ (i = 1, 2, 3, 4) of each wheel is defined as

$$\gamma_i = 1 - \varepsilon_i = \sqrt{\frac{F_{xi}^2 + F_{yi}^2}{F_i}}, \quad \ldots \ldots \ldots \ldots (6)$$

where, $F_{xi}, F_{yi}$: longitudinal and lateral force of the i-th wheel, $F_i$: the radius of the friction circle of the i-th wheel. In this study, the μ rate of each wheel is controlled so as to have the same value as $\gamma_i = \gamma_j \quad (i, j = 1, 2, 3, 4)$ \ldots \ldots \ldots \ldots (7)

Furthermore, the angle between the direction of the friction force and the X-axis is expressed as $q_i$ (i = 1, 2, 3, 4) as shown in Fig. 4. In the second layer of hierarchical control, the value of $q_i$ that minimizes $\gamma$ by satisfying the following constraints is calculated by using Sequential Quadratic Programming (SQP).

$$\gamma \sum_{i=1}^{4} F_i \cos q_i = F_{x0} \quad \ldots \ldots \ldots \ldots (8)$$
$$\gamma \sum_{i=1}^{4} F_i \sin q_i = F_{y0} \quad \ldots \ldots \ldots \ldots (9)$$
$$\gamma \sum_{i=1}^{4} (-d_i \cos q_i + l_i \sin q_i) = M_{z0} \quad \ldots \ldots \ldots \ldots (10)$$

Eq. (8) represents a constraint indicating that the resultant force in the X-axis direction is the target value $F_{x0}$. Eq. (9) represents a constraint indicating that the resultant force in the Y-axis direction is target value $F_{y0}$. Eq. (10) represents a constraint indicating that the moment around the center of gravity of the vehicle is the desired yaw moment $M_{z0}$. 

By eliminating $\gamma$ from Eqs. (8)-(10), the constraints of $q_i$ can be obtained.

$$\sum_{i=1}^{4} F_i \left\{ (-F_{x0} l_i - M_{z0}) \cos q_i + F_{x0} l_i \sin q_i \right\} = 0 \quad \ldots \ldots \ldots \ldots (11)$$
$$\sum_{i=1}^{4} F_i \left\{ (-F_{y0} d_i - F_{y0} l_i - M_{x0}) \sin q_i - F_{y0} d_i \cos q_i \right\} = 0 \quad \ldots \ldots \ldots \ldots (12)$$

The performance function $J$ that minimizes $\gamma$ is defined as

$$J = \frac{(d_0 F_{x0})^2 + (l_0 F_{x0})^2 + M_{z0}^2}{\gamma} \quad \ldots \ldots \ldots \ldots (13)$$

where, $d_0, l_0$: constants of a typical moment arm. In this paper, we use the following constants.

$$d_0 = \frac{T_r + T_s}{4} \quad \ldots \ldots \ldots \ldots (14)$$
\[ l_0 = \frac{L_f + L_s}{2} \]  

Since the numerator on the right side of Eq. (13) is a constant, maximizing \( J \) implies minimizing \( \gamma \). By substituting Eqs. (8)-(10) into Eq. (13), \( J \) can be rewritten as,

\[
J = d_0^2F_{0l} \sum_{i=1}^{4} F_i \cos q_i + l_0^2F_{0l} \sum_{i=1}^{4} F_i \sin q_i \\
+ M_0 \sum_{i=1}^{4} F_i \left( -d_{\phi_0} \cos q_i + l_1 \sin q_i \right) \\
= \sum_{i=1}^{4} F_i \left\{ \left( d_0^2F_{0l} - d_{\phi_0} \right) \cos q_i + \left( l_0^2F_{0l} + l_1 \right) \sin q_i \right\} .
\]  

Then, the optimization problem can be formulated as follows.

**Problem 1:**

We need to find \( q_i \), \( i = 1, 2, 3, 4 \), that maximizes the performance function (16) with constraint equations (11) and (12).

Problem 1 can be solved by using SQP. First, \( \sin q_i \) and \( \cos q_i \) can be linearized around \( q_{i0} \) as:

\[
\sin q_i = \sin q_{i0} + \cos q_{i0} (q_i - q_{i0}) \\
\cos q_i = \cos q_{i0} + \sin q_{i0} (q_i - q_{i0}).
\]

Then, constraint equations (11) and (12) can be expressed as follows.

\[
\sum_{i=1}^{4} F_i \left\{ F_{0l} d_i + M_{i0} \right\} \sin q_{i0} + F_{0l} \cos q_{i0} \right\} q_i \\
= \sum_{i=1}^{4} F_i \left\{ F_{0l} d_i + M_{i0} \right\} \left( \sin q_{i0} + \cos q_{i0} \right) \\
+ F_{0l} \left\{ \cos q_{i0} \cos q_{i0} \right\} .
\]

Next, \( \sin q_i \) and \( \cos q_i \) can be approximated around \( q_{i0} \) by a 2nd order Taylor expansion as:

\[
\sin q_i = \sin q_{i0} + \cos q_{i0} (q_i - q_{i0}) - \frac{\sin q_{i0}}{2} (q_i - q_{i0})^2 \\
\cos q_i = \cos q_{i0} + \sin q_{i0} (q_i - q_{i0}) - \frac{\cos q_{i0}}{2} (q_i - q_{i0})^2
\]

Then, performance function (16) can be expressed as

\[
J = \sum_{i=1}^{4} F_i \left\{ -\frac{1}{2} \left( d_0^2F_{0l} - d_{\phi_0} \right) \cos q_{i0} + \left( l_0^2F_{0l} + l_1 \right) \sin q_{i0} \right\} q_i \\
+ \left\{ d_0^2F_{0l} + l_1M_{i0} \right\} \left( \sin q_{i0} \right) \\
+ \left\{ d_0^2F_{0l} - d_{\phi_0} \right\} \left( 1 - \frac{q_{i0}^2}{2} \right) \cos q_{i0} + \frac{q_{i0}^2}{2} \sin q_{i0} \\
+ \left\{ l_0^2F_{0l} + l_1M_{i0} \right\} \left( 1 - \frac{q_{i0}^2}{2} \right) \sin q_{i0} - q_{i0} \cos q_{i0} \\
= \sum_{i=1}^{4} F_i \left\{ -\frac{1}{2} X_{i0} (q_i - X_i)^2 + Y_i \right\}.
\]

where,

\[
X_i = \frac{X_{i0}}{X_{i0}},
\]

\[
X_{i0} = \left( d_0^2F_{0l} - d_{\phi_0} \right) \left( q_{i0} \right) \sin q_{i0} + \frac{\cos q_{i0}}{2} \sin q_{i0} \\
+ \left( l_0^2F_{0l} + l_1M_{i0} \right) \left( q_{i0} \right) \sin q_{i0} + \frac{\cos q_{i0}}{2} \sin q_{i0} \\
= \sum_{i=1}^{4} F_i \left\{ -\frac{1}{2} X_{i0} (q_i - X_i)^2 + Y_i \right\}.
\]

Assuming that \( X_{i0} > 0 \), the problem of maximizing \( J \) can be rewritten as the problem minimizing

\[
K = \sum_{i=1}^{4} p_i^2 
\]

by the variable transformation

\[
p_i = \sqrt{X_{i0}} \left( q_i - X_i \right).
\]

Further, linearized constraint equations (19) and (20) can be expressed as

\[
\left[ A_{11} A_{12} A_{13} A_{14} \right] p = \left[ B_1 \right].
\]

where,

\[
A_{1i} = \sqrt{X_{i0}} \left( \left( F_{0l} d_i + M_{i0} \right) \sin q_{i0} + F_{0l} \cos q_{i0} \right), \\
A_{2i} = \sqrt{X_{i0}} \left( \left( F_{0l} d_i + M_{i0} \right) \sin q_{i0} + \cos q_{i0} \right), \\
B_i = \sum_{i=1}^{4} F_i \left\{ \left( F_{0l} d_i + M_{i0} \right) \left( q_{i0} - X_i \right) \sin q_{i0} + \cos q_{i0} \right\}.
\]
\[B_z = \sum_{i=1}^{4} F_j \left[ F_{ji} d_i \left\{ (q_{io} - X_i) \sin q_{io} + \cos q_{io} \right\} + \left( F_{ji} l_i - M_{io} \right) \right] \left\{ (q_{io} - X_i) \cos q_{io} - \sin q_{io} \right\} \]. \hspace{1cm} (34)

Since minimizing \(K\) implies minimizing the Euclidian norm of \(p = [p_1 \ p_2 \ p_3 \ p_4]^T\), the optimal solution can be derived by using pseudo-inverse matrix calculation as

\[p = A^{-1} B \hspace{1cm} \text{where, } A^+ \text{ represents the pseudo-inverse of matrix } A.\hspace{1cm} \text{ (35)}\]

Then, \(q = diag \left[ \frac{1}{\sqrt{F_1 X_{d1}}} \ \frac{1}{\sqrt{F_2 X_{d2}}} \ \frac{1}{\sqrt{F_3 X_{d3}}} \ \frac{1}{\sqrt{F_4 X_{d4}}} \right] \)

\[\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \hspace{1cm} \text{ (36)}\]

where, \(q = [q_1 \ q_2 \ q_3 \ q_4]^T\).

Then, recursion formula solving of optimal \(q_i\) based on SQP is shown as follows.

**STEP 1** To satisfy the condition \(X_{d1i} > 0\), the initial value of \(q_{io}\) is calculated as

\[q_{io} = \tan^{-1} \left( \frac{l_i^2 F_{io} + l_i M_{io}}{d_i^2 F_{io} - d_i M_{io}} \right) \hspace{1cm} \text{ (37)}\]

By substituting Eq. (37), \(X_{d1i}\) can be rewritten as

\[X_{d1i} = \left( d_i^2 F_{io} - d_i M_{io} \right) \cos q_{io} + \left( l_i^2 F_{io} + l_i M_{io} \right) \sin q_{io} \]

\[= \sqrt{\left( d_i^2 F_{io} - d_i M_{io} \right)^2 + \left( l_i^2 F_{io} + l_i M_{io} \right)^2} \cos \left( q_{io} - \tan^{-1} \left( \frac{l_i^2 F_{io} + l_i M_{io}}{d_i^2 F_{io} - d_i M_{io}} \right) \right) \]

\[= \sqrt{\left( d_i^2 F_{io} - d_i M_{io} \right)^2 + \left( l_i^2 F_{io} + l_i M_{io} \right)^2} > 0. \hspace{1cm} \text{ (38)}\]

**STEP 2** \(q_i\) is calculated from Eq. (36).

**STEP 3** We define a penalty function as

\[P(q) = \frac{1}{J_{F_i}(q)} + \rho \left( \left| J_{F_i}(q) \right| + \left| J_{F_i}(q) \right| \right) \hspace{1cm} \text{ (39)}\]

where, \(\rho > 0, \ J_{F_i}(q) = \sum_{i=1}^{4} F_i \left\{ (-F_{io} d_i - M_{io}) \cos q_i + F_{io} l_i \sin q_i \right\}, \hspace{1cm} \text{ (40)}\]

\[
J_{F_i}(q) = \sum_{i=1}^{4} F_i \left\{ -F_{io} d_i - M_{io} \right\} \cos q_i + \left( F_{io} l_i \right) \sin q_i \}
\]

\[
J_{F_i}(q) \text{ corresponds to the left side of Eq. (11), and } J_{F_i}(q) \text{ corresponds to the left side of Eq. (12). If } P(q) < P(q_0), \text{ then }
\]

\[q_0 - q \hspace{1cm} \text{ (42)}\]

and we can go to STEP 2. Otherwise, \(\hat{q}\) which satisfies \(P(\hat{q}) < P(q_0)\) is a straight line searched in \([q, q_0]\). Then,

\[q_0 = \hat{q} \hspace{1cm} \text{ (43)}\]

and we can go to STEP 2.

We know that SQP is one of the most effective methods of nonlinear optimization. Further, optimal \(\gamma\) can be obtained as

\[
\gamma = \frac{1}{\sum_{i=1}^{4} F_i \left\{ (d_i^2 F_{io} - d_i M_{io}) \cos q_i + (l_i^2 F_{io} + l_i M_{io}) \sin q_i \right\}} \hspace{1cm} \text{ (44)}
\]

The magnitude and direction of each tire force are translated to the steer angle and longitudinal force of each wheel, and these are controlled in the last layer of the hierarchical control. The lateral and longitudinal forces can be obtained as

\[
F_{ai} = \gamma F_i \cos q_i \hspace{1cm} \text{ (45)}
\]

\[
F_{ai} = \gamma F_i \sin q_i \hspace{1cm} \text{ (46)}
\]

Further, by using a brush model, the slip angle of each wheel can be calculated as

\[
\alpha_i = \tan^{-1} \left( \frac{K_{ai} - \kappa \sin q_i}{K_{ai} (1 - \kappa \cos q_i)} \right) \hspace{1cm} \text{ (47)}
\]

where,

\[
\kappa = \frac{3F_i}{K_{ai} \left[ 1 - (1 - \gamma)^2 \right]} \hspace{1cm} \text{ (48)}\]

\(K_{ai}: \text{ braking stiffness }, K_{ai}: \text{ cornering stiffness.}\)

### 4. Simulation results

**Figure 5** shows the simulation results for four-wheel distributed steering and traction braking control in the case of a single-shot sine-wave steering maneuver at \(v = 22 \text{ m/s}\). A linear model of the vehicle is used for the reference of control. The \(\mu\) rate is controlled such that it is the same for each wheel, and is restricted to a maximum value of 0.95, so that the grip margin is uniform, while the limit on the friction circles is satisfied. This implies that the
tire forces are used efficiently in the friction circles. Figure 6 compares the maximum resultant force of the simulation result for a single-shot sine-wave steering maneuver at \( v = 22 \text{ m/s} \). This figure shows that the limited performance (maximum resultant force) is improved through the use of integrated control. The maximum resultant force is improved by 7.8% as a result of adding active front-rear steering control to four-wheel distributed traction/braking control, and by a further 2.5% as a result of adding an independent steering mechanism (four-wheel distributed steering).

5. Conclusion

In this paper, the radius of each tire friction circle is estimated from the relationship between the Self-Aligning Torque (SAT) and the lateral and longitudinal forces acting on each tire, and a method of vehicle dynamics control that uses the tire friction circle of each wheel for the optimum is proposed. A distribution algorithm using Sequential Quadratic Programming (SQP) is used to calculate the magnitude and direction of the tire forces, such that they satisfy the constraints corresponding to the target force and moment of the vehicle motion and minimize each tire \( \mu \) rate. The proposed algorithm, being characterized by pseudo-inverse matrix calculation, features high-speed calculation and accurate calculation that satisfy the constraints, so that it can be applied to four-wheel distributed steering and four-wheel distributed traction/braking systems for which the optimization of the eight parameters is necessary. Then, the effectiveness of the vehicle dynamics control for four-wheel distributed steering and four-wheel distributed traction/braking systems is shown by simulations.

References


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